Transcendental syntax III: equality

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February 25, 2018

Donner un sens plus pur aux mots de la tribu :
à P.-L., ami et compagnon de route de 30 ans.

Abstract

Revisiting a neglected area, logical equality, we eventually reduce predicates to propositions.

1 Introduction

One should never say that the King is naked. But, like it or not, predicate calculus is a half-baked contraption, lame from the logical standpoint, not even able to hande equality as a logical primitive.

1.1 Axiomatic realism

Predicate calculus is nevertheless satisfactory as an *axiomatic* system; but axiomatics is the poor man's logic. This approach, dominant before Gentzen, handled logical operations through schemas. For instance $A \Rightarrow (B \Rightarrow A)$ — indeed the second order axiom $\forall X \forall Y(X \Rightarrow (Y \Rightarrow X))$ in disguise.

In logical terms, axiomatics remains an efficient way to obtain *negative* results: think of incompleteness, formulated in terms of rather arbitrary axiomatic systems. It is surely a useful *reduction*, but beware of reductionism! Axiomatics — the greek *axiomatikos* means « officer » — is indeed the dictatorial approach to reasoning; while logic should be natural, surely not enforced by martial law. The identification of logic with axiomatics is therefore a misunderstanding.

Predicate calculus, created at a time when logic was still reduced to axiomatics, bears some stigmata of this misconception. For instance, the logical handling of the universal quantifier $\forall x$ involves the use of generic variables, a.k.a. *eigenvariables* « choose an arbitrary x »; axiomatics forgot the generic origin of variables and used them as if proceeding from the sky. With some collateral damages, typically the familiar principle $\forall x A \Rightarrow A[t/x]$ and its dual form $A[t/x] \Rightarrow \exists x A$ yield the faulty consequence $\forall x A \Rightarrow \exists x A$. This proof relies on the term t whose use is granted by the axiomatic introduction of non generic variables: let $t := y$. The mistake can be fixed by making y generic, e.g., $\forall y(\forall xA \Rightarrow \exists xA)$, but the « naked » $\forall xA \Rightarrow \exists xA$ remains faulty.

Another misconception rests upon semantics, yet another reduction whose role — the production of counterexamples — is basically negative: again, beware of reductionism! Logic is indeed the expression of a distrust of this « reality » which is often another name for « prejudice »: the logician is not satisfied with eating his cake until he knows why!

If axiomatics and semantics do — separately — play interesting roles in logic, their combination, *axiomatic realism*, is a sort of criminal association. Typically, the faulty principle $\forall xA \Rightarrow \exists xA$, a mistake of the Army of Aaxiomatics is justified by the Church of Reality: the notion of model has been doctored so as to exclude empty domains. The use of illogical individuals — non generic variables — is compensated by the exclusion of some part of the reality. The alleged reason is that the empty model is not that exciting; as if Justice were dismissing a witness on the grounds that he has presumably nothing to say!

The mistake $\forall xA \Rightarrow \exists xA$ remained unnoticed for pragmatic reasons: around 1900, logic was assigned the menial task of formalising mathematics — which involved an axiomatic approach and nothing more. And this dubious principle, although logically faulty, holds anyway for external, axiomatic reasons.

But this casts a doubt as to the logical value of predicate calculus: logic should be « zero defect[1] »; at least if it wants to be styled « logic »!

1.2 Individuals

We are now far from the turn of the XX$^{\text{th}}$ century and no longer obsessed with foundations. Constructivism and, more recently, computer science prompted the study of logic *for itself*. A lot of progress has been made as to propositions, logical connectives. But quantification, especially first order, has been left aside.

Indeed, first order quantification seems a bit superfluous. The « forgetful functor » which erases individuals — as well as the quantifiers $\forall x$ and $\exists x$ —, thus replacing any predicate $P(t_1, \ldots, t_n)$ with a proposition P, preserves almost everything. To the point that there is nothing more in cut-elimination for full second order predicate logic than for system $\mathbb{F}$ which can be seen as its forgetful image: in algorithmic terms, first order individuals and quantifiers bring strictly nothing.

Strangely enough, the forgetful functor stumbles on equality: we cannot replace $t = u$ with a proposition $=$. The problem is fixed in a rather *ad hoc* way, realisability. Typically, $t = u$ becomes a proposition **bool** with two elements **t**,**f**; an external comment explaining whether or not a propositional proof « realises » a formula is now needed. This semantic contraption works at the price of a total confusion. This is why the standard formulation of predicate calculus excludes equality from the list of logical primitives: it handles it as a predicate among others.

[1]The analytic metaphysicist Quine based his ontology upon the existential quantifier of predicate calculus. The idea of reducing a rather delicate — not to say fishy — subject like ontology to logic is quite suspect; but *quid* of reducing it to a lame system?

This mistreatment of equality by predicate calculus is a strategic failure, for the simple reason that the ultimate predicate is precisely equality. If you are not convinced, open any textbook and observe the pregnancy of equations, commutative diagrams, etc. By the way, the work expounded here didn't originate in a radical distrust of predicate calculus, but in the desire to find a satisfactory status for what should be the main operation of logic and is handled axiomatically, i.e., as an arbitrary predicate. To the point that nothing serious on the subject has been said since... Leibniz. Starting from his half-baked approach, we shall eventually find the surprising solution to our quest for equality: in $t = u$, t and u should be treated as propositions, in which case equality becomes the *linear* equivalence $t \equiv u$.

The major logical prejudice concerns truth values: we are convinced that — in some ideal semantic paradise — everything is true or false. What is usually expressed by the excluded middle $a \vee \neg a$, a formulation which involves the most complex classical negation. I prefer the lighter $a \equiv b \vee b \equiv c \vee c \equiv a$, which uses *linear* equivalence $\equiv$: among three propositions, two are equivalent. The idea of attaching a truth value to a property is however questionable: think of `white` which, being a quality, cannot receive any truth value. Logical realism solved the problem by means of a monstrous contraption, namely the introduction of *individuals* whose only role is to replace properties like `white`, without truth values, by predicates of the form $\texttt{white}(x)$ « x is white » which may now be true of false. First order predicates and individuals are thus a way to comply with our prejudice concerning truth values: in the name of reality, we created monsters whose logical status is hard to ascertain. And these monsters now want to be fed...

This reminds us of the geocentric prejudice and the way Ptolemy handled the « backward motion » of Saturn: planets were supposed to be equipped with *epicycles*, sort of fantasmatic little wheels. In this light, the individuals of predicate calculus are the epicycles of modern logic — with Frege playing the part of Ptolemy.

2 Equality

The problems connected with equality are a collateral damage of the introduction of first order individuals — the metaphysical terms and and variables. There is no need for those, since they can be identified with certain propositions. Provided we handle them in a linear way: equality becomes linear equivalence $a \equiv b$. Linear logic is eventually justified as the only approach which does not mistreat equality.

2.1 Leibniz

Natural numbers and equality are the most basic notions of mathematics; however, after one century of proof-theoretic investigations, we must agree that integers are definitely a second order — i.e., a very complex — notion. But *quid* of equality? It has been so neglected that, after 300 years, Leibniz's definition

$$a = b \quad :\Leftrightarrow \quad \forall X(X(a) \Rightarrow X(b))$$

is still in use: typically, in first order disguise, through the axomatic schema

$$(a = b \wedge A[a/x]) \Rightarrow A[b/x]$$

Leibniz's definition, which refers to all possible predicates $X(\cdot)$, makes equality a most complex, super-synthetic contraption. While it should remain simple, almost analytic; in particular quantifier-free.

The worse is still to come: the definition is almost empty. Typically, are the two « `n` » in « `meaning` » equal? Since one is to the right of the other, they can be distinguished by a property $P(\cdot)$. The question thus reduces to the *relevance* of $P(\cdot)$: Leibniz's equality supposes a *preformatting* which decretes what we can/cannot consider as a legitimate property of an individual. And « legitimate properties » eventually turn out to be those compatible with... equality!

On this issue, semantics plays its usual role of bribed witness: the two « `n` » are equal when they refer to the same ideal individual, i.e., when we decide to identify them!

Indeed, the basic mistake seems to originate in the very notion of individual: after all, why should there be individuals? The only justification lies between convenience — the expression of algebraic structures — and pure conservatism: any discussion of the relevance of « individuals » is a sort of blasphem against Frege. But no serious logical argument, as we shall now prove by reducing them to specific propositions.

As expected, the answer lies in proof-nets. Let us try to write proof-nets for Leibniz's equality $\forall X(X(a) \multimap X(b))$. We first remove the quantifier $\forall X$ and we are left with the linear implication $X(a) \multimap X(b)$. Now, when we use links between « occurrences » of $X(a)$, $\sim X(a)$, the X hardly plays any role: we could as well directly link a and $\sim a$. In which case a becomes a proposition, and equality reduces to linear equivalence:

$$a \equiv b \quad :\Leftrightarrow \quad (a \multimap b) \;\&\; (b \multimap a)$$

Indeed, Leibniz's montrous quantification collapses to the two cases: $X(a) := a$ and $X(a) := \sim a$. Provided we can handle individuals as ordinary propositions!

If terms are indeed propositions, the aporia concerning the two `n` disappears: we are not relating properties of two individuals, we are just relating two properties, period; the question of the relevance of $P(\cdot)$ thus disappears.

2.2 Enough individuals

We simplified Leibniz's definition by forgetting X in $\forall X(X(a) \equiv X(b))$. This X is a typical epicycle, due to the classical prejudice concerning truth values:

Every proposition is eventually true or false.

Classical logic forbids the identification of individuals with propositions because of the tautology $a \equiv b \vee b \equiv c \vee c \equiv a$: there would be at most two propositions. Ditto with intuitionistic logic which complies with the prejudice, doubly negated as $\neg\neg(a \equiv b \vee b \equiv c \vee c \equiv a)$.

The situation is quite different for linear logic, where infinitely many distinct individuals can coexist: $a_i \not\equiv a_j$ for $i \neq j \in \mathbb{N}$. This is the ideal place where semantics may play a role: consider as phase space the additive group $\mathbb{Z}$ with $\perp := \mathbb{Z} \setminus \{0\}$. If $X \subset \mathbb{Z}$, then $\sim X = \{-x; x \notin X\} = -(\mathbb{Z} \setminus X)$, hence all subsets of $\mathbb{Z}$ are facts. Among them, $\mathbf{0} := \emptyset$, $\mathbf{1} = \{0\}$; exponentiation is defined by $!X := X \cap \{0\}$. (Intuitionistic) negation $\neg X := !X \multimap \mathbf{0}$ is thus $X \cap \{0\} \multimap \emptyset$: $\neg X = \mathbf{0}$ if $0 \in X$, $\neg X = \top = \mathbb{Z}$ otherwise. If X, Y are distinct, then 0 cannot belong to both $X \multimap Y$ and $Y \multimap X$, hence, $\neg(X \multimap Y \,\&\, Y \multimap X)$ takes the value $\top$. All facts of the space are therefore *logically* distinct, in the strong sense that their inequivalence takes the value $\top$.

The replacement of a with $X(a)$, of $\sim a$ with $\sim X(a)$ is a way of instilling some linearity in a classical or intuitionisitic setting: the contraction rule does no longer apply to a. $X(\cdot)$ is used like a modality, a sort of condom interrupting the flow of logical consequence. This interruption is made necessary by the realistic prejudice excluding the linear maintenance of propositions. As a collateral damage, realism created epicycles: individuals and their predicates.

2.3 Terms as multiplicatives

We so far reached a rather interesting hypothesis, equality as (linear) equivalence. This hypothesis is backed by the fact that equality should not be axiomatic, i.e., arbitrary, but natural, logical. The so-called individuals thus become propositions; the good news is that, provided we keep a linear maintenance of those individuals, there may be enough of them.

Now, the identification « individual = proposition » is faulty: even if predicate calculus is lame, the distinction between first order and second order has a technical contents in terms of, say, subformulas. Therefore, individuals should not correspond to all propositions, but only to those of a « simple » form. In view of our general definition [5] of n-ary multiplicatives ($n > 0$), we propose to identify individuals with multiplicative propositions.

Now, there is something delicate to understand, linked to the main prejudice engraved in us by predicate calculus, that of the « domain » of interpretation. Individuals are multiplicative, yet we cannot name a single one. This is due to the impossibility of 0-ary multiplicatives. To sum up, there is nothing like a definite individual: individuals make only sense as parametric expressions (terms) depending upon variables. This is consistent with the aforementioned failure of $\forall x A \Rightarrow \exists x A$ — definitely a mistake.

Terms (= parametric individuals) are therefore obtained from variables $\alpha, \beta \ldots$ by means of multiplicative connectives. The typical term is therefore $\mathcal{C}[\ldots \alpha_i \ldots]$, where $\mathcal{C}$ is of arity I and $\alpha_i (i \in I)$ are variables, not necessarily distinct; their negations are not allowed. For instance, if ¶ is the 4-ary connective consisting of $\{\{1,2\},\{3,4\}\}$ and $\{\{2,3\},\{4,1\}\}$, $\rho := \P[\alpha, \beta, \beta, \gamma]$ is a term. Terms can easily be composed using the definition given in [5]; we must of course take care of repetitions of variables, e.g., the two β in ρ.

Equality is linear equivalence $\rho \equiv \sigma$. If $\rho = \mathcal{C}[\ldots\alpha_i\ldots]$ and $\sigma = \mathcal{D}[\ldots\beta_j\ldots]$, equality reduces to the existence of two bijections φ, ψ s.t. $\alpha_i = \beta_{\varphi(i)} (i \in I)$ and $\beta_j = \alpha_{\psi(j)} (j \in J)$ s.t. $\mathcal{C} \subset_\varphi \mathcal{D}$, $\mathcal{D} \subset_\psi \mathcal{C}$. As observed in [5], this makes $\mathcal{C}$ and $\mathcal{D}$ isomorphic. Because of possible repetitions of variables, the choice of φ and ψ is not always unique, in particular ψ need not be the inverse of φ. In presence of repetitions, a proof of equality has therefore a non trivial contents, namely the two unrelated bijections φ and ψ.

2.4 Injectivity

A last problem should be adressed before we can be sure we really solved the problem: that of function letters. We excluded the 0-ary case, the so called constants which should be handled as derelict variables, i.e., universally quantified: the faulty $\forall\alpha A \Rightarrow \exists\alpha A$ is thus fixed into $\forall\beta(\forall\alpha A \Rightarrow \exists\alpha A)$. But function letters of positive arity do make sense.

Binary function letters have a well-known property of *injectivity*, which allow their use as pairing functions. Technically speaking, this means that, whenever $f(u, v) = f(u', v')$ is provable, then $u = u'$ and $v = v'$ are provable as well. We therefore need a term $\tau[\alpha, \beta]$ such that, whenever $\tau[\sigma, \rho] \equiv \tau[\sigma, \rho]$ is provable, then $\sigma \equiv \sigma'$ and $\rho \equiv \rho'$ are provable as well.

Such a pairing function is provided by $\tau[\alpha, \beta] := (\alpha \parr \beta) \otimes (\alpha \parr \alpha \parr \beta)$, an example losely inspired from that of set theory $(x, y) := \{\{x\}, \{x, y\}\}$. Assume that $\tau[\sigma, \rho] \equiv \tau[\sigma, \rho]$; this means that the two connectives underlying $\tau[\sigma, \rho]$ and $\tau[\sigma, \rho]$ are isomorphic. An easy lemma shows that any connective can be split as the $\parr$ of several connectives that cannot be split further; moreover, this splitting is unique up to permutation. A similar splitting holds, dually, for $\otimes$, hence $(\sigma \parr \rho) \otimes (\sigma \parr \sigma \parr \rho)$ can be uniquely split as a tensor $\mathtt{T} \otimes \mathtt{U}$. Both of $\mathtt{T}, \mathtt{U}$ can in turn be split into a $\parr$ of prime components $\tau_i\,(i = 1, \ldots, k)$; this decomposition is unique if we consider the τ_i up to equivalence and introduce their multiplicities m_i and n_i in $\mathtt{T}, \mathtt{U}$; $\mathtt{T}, \mathtt{U}$ can be distinguished by the requirement $m_i \leq n_i\,(i = 1, \ldots, k)$. We can recover σ as the $\parr$ of the τ_i each of them with multiplicity $n_i - m_i$ and ρ as the $\parr$ of the same τ_i, each of them with multiplicity $2m_i - n_i$.

2.5 Natural numbers

We addressed all questions related to equality in pure logic. However, important systems use additional proper axioms for equality. This is the case for Peano's arithmetic, based on a second order principle — recurrence, reformulated as a first order schema — and various first order axioms, typically:

$$Sx \neq 0$$
$$Sx = Sy \Rightarrow x = y$$

These axioms should be taken seriously, but not literally; for instance, they involve a constant 0 which cannot make sense logically speaking. They have been written in a spirit of axiomatic expediency: *something like that* is needed, but perhaps in an alternative formulation. However, even if miswritten, these axioms do imply the existence of denumerably many distinct individuals $\mathtt{m} \neq \mathtt{n}$. We already observed that this existence holds in a specific phase model, i.e., axiomatically; but *quid* of its logical validity?

Since linear logic can be « strengthened[2] » by means of weakening and contraction, then $\mathtt{m} \neq \mathtt{n}$ cannot be logically valid, since it would still hold classically. But this only applies to sequential operations, those which can be formulated in sequent calculus. But not to general, non sequential logical operations like the 4-ary ¶ of [5].

The question at stake is therefore to find the right notion of individual enabling us to *prove* all first order Peano axioms — or something losely equivalent. As we just explained, the solution would be anti-classical, i.e., classically inconsistent. On the other hand, since recurrence is basically an instance of second order quantification, arithmetic would become a logical — axiom-free — system.

3 Derealism and epidictics

3.1 Derealism

Dedekind's definition of $\mathbb{N}$ « the smallest set containing 0 and closed under S »

$$n \in \mathbb{N} \quad :\Leftrightarrow \quad \forall X\, (\forall x\, (X(x) \Rightarrow X(Sx)) \Rightarrow (X(0) \Rightarrow X(n)))$$

involves a second order quantification which is responsible for the monstrous expressive power of logic; and its major limitations as well. Forgetting first order, we get $\mathbf{nat} : \forall X((X \Rightarrow X) \Rightarrow (X \Rightarrow X))$, the familiar definition of natural numbers in system $\mathbb{F}$.

The sequent $\mathbf{nat} \vdash (B \Rightarrow B) \Rightarrow (B \Rightarrow B)$ is a way of expressing *iteration*: if the left hand side is fed with a natural number n, the right hand side becomes the functional $\Phi(f) := \overbrace{f \circ f \circ \ldots \circ f}^{n \text{ times}}$. Natural numbers and iterators can be logically constructed with the help of the second order links:

$$\frac{A[B/X]}{\exists X A} \qquad\qquad \frac{A}{\forall X A}$$

Consider iterators: if $B^4 := (B \Rightarrow B) \Rightarrow (B \Rightarrow B)$, let π be the proof-net with conclusions $\sim B^4, B^4$. Second order existence enables us to pass from $\sim B^4$ to $\exists X \sim X^4 = \sim\mathbf{nat}$, thus yielding a proof-net π_B of conclusions $\sim\mathbf{nat}, B^4$ expressing iteration. Now, π is correct since it complies with the ordeals associated with $\sim B^4$ and B^4, but what about π_B?

[2] Or castrated, it's a matter of viewpoint.

There is no difference, except that the divide Object/Subject is no longer respected: this is what we style *derealism*. This divide was at work in the opposition vehicle/ordeal: on one side, the analytic « reality », on the other side, the synthetic subjectivity. In the case of π_B — more generally of second order existence —, no way of guessing the « witness » B, hence we cannot find any ordeal corresponding to $\exists X A$. This witness is indeed a gabarit; we can, however, check correctness, provided this ordeal becomes part of the data, i.e., part of the « Object ». In other terms, when we deal with second order, three partners are involved: a vehicle $\mathcal{V}$, an ordeal $\mathcal{O}$ and a subsidiary gabarit — a *mould* — $\mathcal{M}$ corresponding to existential « witnesses »: we require the strong normalisation of $\mathcal{V}+\ \mathcal{O}+\mathcal{M}$ into $[\![\, p_\Gamma(x) \,]\!]$. The opposition is now between $\mathcal{O}$ and $\mathcal{V}+\mathcal{M}$, i.e., between the Subject and a compound Object/Subject, an *épure*[3].

Technically speaking, little has so far changed: there is no essential difference between $\mathcal{V}$ and $\mathcal{V}+\mathcal{M}$. But the spirit is new, in particular the relation to *l'usage* becomes far more complex. Let us explain: in an opposition $\mathcal{V}/\mathcal{O}$, everything depends upon the mathematical properties of $\mathcal{O}$, no hypothesis on $\mathcal{V}$ has to be made; in an opposition $(\mathcal{V}+\mathcal{M})/\mathcal{O}$, part of *l'usage* will depend upon properties of the mould $\mathcal{M}$ which cannot really be checked.

Take, for instance, the iterator π_B: second order existence asks us for this witness B which occurs four times in $\sim B^4$, respectively negatively, positively, positively, negatively. It thus requires four gabarits, two for B, two for $\sim B$; the two positive ones should be « the same », ditto for the negative ones. Moreover the positive and negative gabarits should be « complementary », the negation of each other. Complementarity of gabarits $\mathcal{G}, \mathcal{G}'$ basically means that we can perform (i.e., eliminate) a cut between them.

In other terms, the choice of the mould $\mathcal{M}$ postulates a reduction *usage/usine* that does not belong in any decent analyticity. Here lies the very source of foundational doubts: the Object, seen as an épure, embodies in itself something beyond justification.

The need for this auxiliary part should be obvious from the limitations of the category-theoretic approach to logic. In terms of categories, the absurdity $\mathbf{0}$ is an initial object, with the consequence that we cannot distinguish between morphisms into this object. Less pedantically, the absurdity is the empty set, and they are too few functions with values into $\emptyset$ to make any useful distinction between them: either $A = \emptyset$ and there is only one function from A to $\emptyset$ or $A \neq \emptyset$ and there is none. As a consequence, negation $\neg A := A \Rightarrow \mathbf{0}$ becomes a bleak operation: from the functional, category-theoretic standpoint, $\neg A$ is either empty or with a single element.

Now, remember that many mathematical results or conjectures are negations: « there is no solution to… ». According to the category-theoretic prejudice, the intricate proofs of these results are but constructions of the function from $\emptyset$ to itself…which is quite preposterous! It would be more honest to ad-

[3]This word, without satisfactory translation, refers to the representation of an object (our vehicle) through several viewpoints (those of the mould).

mit that we reach here one of the major blindspots of logic. This blindspot is sometimes styled as « a proof without algorithmic contents ».

The notion of épure could explain the situation as follows: the vehicle in a proof $\mathcal{V}+\mathcal{M}$ of a negation may be trivial ($\mathcal{V}=0$). A major part of the real proof is the mould $\mathcal{M}$ which makes no sense in category-theoretic terms, but which may be very intricate.

As to typing, there are two opposite approaches, those of Curry and Church: for Curry, the objects are born untyped, the typing occurs later. To this sort of *existentialism*, Church opposes an *essentialism* for which objects are born with their types. The opposition vehicle/ordeal is a sort of implementation of typing à la Curry; such an attempt was necessary for the sake of — say — rationality. Second order shows that Church is not that wrong; not quite that objects are actually born with their types, but that they embody some typing in them through the mould $\mathcal{M}$. But the épures $\mathcal{V}+\mathcal{M}$ must be opposed to $\mathcal{O}$ to get their type, i.e., they are still untyped. Finally, the derealism at work in *épures* reconciles the two viewpoints.

3.2 Discussion: second order

Second order is considered as suspect, witness *predicativity*: the notion was introduced by Poincaré, the greatest mathematician of his day, but surely not an expert on logic which he cordially hated. If one can hardly blame him for not taking seriously the foundational crisis, he should have been consistent and shut up on the subject rather than proposing a flippant solution.

First order logic involves certain second order features, typically through the so-called predicate constants $P(\cdot), Q(\cdot)$, etc. Those are indeed universally quantified second order variables: for instance, the theorem $\forall y(P(y) \Rightarrow P(y))$ should indeed be written $\forall X \forall y(X(y) \Rightarrow X(y))$. Due to the implicit presence of the quantifier $\forall X$, it is not possible to negate « first order » formulas: this would require the existential quantifier $\exists X$ which cannot be kept implicit. Negation is thus performed externally: $\exists X$ is translated as « there is a counter-model ». We see that first order involves a partial externalisation of second order features.

Second order can also be reformulated in terms of schematic rules, typically *recurrence*. The principle $\forall X((X(0) \wedge \forall x(X(x) \Rightarrow X(Sx))) \Rightarrow X(n))$ is thus replaced with instances $A[0/x] \wedge \forall x(A \Rightarrow A[Sx/x]) \Rightarrow A[n/x]$. This is nothing but second order in disguise: like or not, arithmetic is a second order system.

Certain logical connectives, although styled first order, are indeed second order. The obvious example is intuitionistic disjunction $\vee$ whose elimination rule is that of $\forall X((A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X))$. The connective is indeed lame: its elimination rule mingles two proofs of $A \Rightarrow C$ and $B \Rightarrow C$ into a proof of $A \vee B \Rightarrow C$ which is indeed a proof of $A \vee B \multimap C$. If there were something like a unary disjunction $\vee A$, it would replace $A \Rightarrow C$ with $\vee A \multimap C$, in other terms $\vee A = !A$: like it or not, intuitionistic disjunction contains the exponential of linear logic which is also problematic, i.e., not first order, see [4]. The only way to fix intuitionistic disjunction — i.e., to keep it first order — is to replace it with the additive $\oplus$ which is not that far from $\vee$. Intuitionistic negation is also a second order notion, see section 4.

To sum up, the scarecrow of second order is often disguised as first order... in the same way the 13^{th} row of planes are often renumbered 14.

First order logic, i.e., propositional calculus, relies upon a healthy opposition between Object — the vehicle of the proof-net — and Subject — the ordeals, a.k.a. switchings. Second order, especially existential quantification, involves a change of paradigm: what is judged by the ordeal is no longer a vehicle, but a combination involving a subjective part — the mould —, i.e., a choice of switchings. The global judgement about the correctness of a proof depends upon a expert — the mould — provided by the proof itself. There is a conflict of interest: as a correctness criterion, the mould should be strict but, since on the part of the defendant, he may be laxist[4]. All foundational questions arise from this ambiguous status of the mould whose trustfulness cannot be tested.

Epidictics is the very choice of those moulds that can be used in second order quantification: it can be styled as the civilised version of axiomatics. The basic point is that the introduction rule $\vdash \forall$ does not determine the elimination $\forall \vdash$ (the stock of possible T for which X can be substituted)[5]. Moreover, even if some are safer than others, no epidictic choice is absolutely safe. The good news is that logic may simultaneously afford several second order quantifications. For instance, second order predicate calculus which corresponds to two sorts of second order quantifiers, « individuals » and « predicates ».

What does this mean in terms of the familiar subformula property? The first order rule $\forall \vdash$ involved substitutions $A[t/y]$. $A[t/y]$ is styled a subformula of $\forall y A$; the notion of subformula has however been doctored to accommodate these substitutions. The excuse is that we can keep some control upon those « subformulas ». A control that collapses in the full second order case: $A[T/X]$ may be more complex than $\forall X\, A$. What is at stake in the subformula property is indeed the « epidictic » choice of the possible substitutions $X \rightsquigarrow T$: make it tame and we shall retain a sort of subformula property, make it too lax and we lose any control. Indeed, when interpreting first-order quantification as a quantification over propositions, only those of a certain kind — multiplicative propositions — are considered. To the effect that the subformula property persists; of course, the notion of subformula adapted to this case is slightly *ad hoc*, but not more than the one used in the old style treatment of first order.

4 Negation

Usual negation $\neg A := A \Rightarrow \mathbf{0}$ is indeed a second order operation: this is due to the use of the constant $\mathbf{0}$ — the neutral element of the disjunction $\oplus$ — which cannot be handled at first order. Its usual rule is the axiomatic schema $\mathbf{0} \Rightarrow A$; in other terms $\mathbf{0} = \forall X\, X$.

[4]Think of Volkswagen whose cars embodied laxist meters.

[5]The same is true in the (old style) first order case: the choice of the substitutions $A[t/y]$ is not determined by the introduction; but this has no dramatic consequence.

4.1 Épures

Since quantification (especially second order) involves a lot of red tape, we shall devise a direct approach to $\mathbf{0}$ and $\top$, our first encounter indeed with the derealist world of épures.

Colours are unary functions $\mathtt{mag}/\mathtt{grn}$, $\mathtt{ylw}/\mathtt{blu}$ and $\mathtt{cya}/\mathtt{red}$. Let us now change — or liberalise — our conventions:

- The rays t and $c(t)$ where t makes no use of the colour functions and c is a colour function are styled *objective*.
- The other rays are styled *subjective*: for instance, $x \cdot \mathtt{cya}(y)$ or $\mathtt{red}(\mathtt{red}(\mathtt{r}))$.

A star is *objective* (resp. *subjective*) when all its rays are objective (resp. subjective) ; a star with both objective and subjective rays is styled *animist*. The result $\mathcal{S}\theta$ of a substitution in a non animist star remains non animist: this is plain if $\mathcal{S}$ is subjective; if $\mathcal{S}$ is objective, depending whether or not some $x\theta$ — with x occurring in $\mathcal{S}$ — is subjective, then $\mathcal{S}$ will be subjective or objective.

An *épure* is a constellation without animist stars. If we want to stress the fact that a constellation may not be an épure, we speak of an *anima* (plural: animæ). The normal form of an épure remains an épure : the actualised dendrite connects various stars, either objective or subjective through common rays $t\theta = u\theta$; if one of those actualised rays happens to be subjective, then connectedness forces all other rays to be subjective as well.

It is convenient to split an épure $\mathcal{C}$ as the sum $\mathcal{V} + \mathcal{M}$ of a *vehicle* $\mathcal{V}$ (its objective part) and a *mould* $\mathcal{M}$ (its subjective part) ; however, since some non trivial coherence may occur between $\mathcal{V}$ and $\mathcal{M}$, $\mathcal{C}$ cannot be recovered from them.

We can see an épure as the analytic form of a typed term of — say — system $\mathbb{F}$. The vehicle corresponds to the underlying pure λ-term whereas the mould takes care of the internal typing at work in the *extractions* (type applications) $\{t\}T$. The possibility for an objective and a subjective star to combine into another (subjective) star accounts for the interaction between terms and types: typically, a typed term $\lambda x^A \cdot t$ will delocate, erase, duplicate all informations (including internal typings) from the location A to the various locations corresponding to the « occurrences » of x in t. In this way, épures manage to reconcile the forgetful functor which erases all informations pertaining to types with the fact that pure terms do interact with types. General animæ — although quite manageable — are considered as logically incorrect because they admit no forgetting. Animæ thus fix the nightmare of « empty types » like $\mathbf{0}$ by providing a class of illegal, illogical inhabitants.

4.2 The additive truth

The additive neutrals — the most basic example of derealism — correspond to second order propositions $\top := \forall X\, X$ and $\mathbf{0} := \exists X\, X$. However, since X occurs only positively, there is no need to match X and its negation $\sim X$, hence no problem of epidictics (section 6). What follows is a direct description of $\top$ and $\mathbf{0}$ without reference to second order quantification.

If $\top(x) := p_\top(x), R(x) := p_\top(\mathtt{c}\cdot x), S(x) := p_\top(\mathtt{mag}(\mathtt{l}\cdot x)), T(x) := p_\top(\mathtt{mag}(\mathtt{r}\cdot x))$, the additive neutral $\top$ is defined by the ordeals $[\![\frac{R(x), S(x)}{}]\!] + [\![\frac{T(x)}{\top(x)}]\!]$ and $[\![\frac{R(x), S(x)}{\top(x)}]\!]$, the latter being *cancelling*, see [4].

The anima $[\![\frac{}{\boxed{R(x)}}]\!] + [\![\frac{}{\boxed{S(x)}, \boxed{T(x)}}]\!]$, which complies with $\top$, splits as a sum $\mathcal{V} + \mathcal{M}$: it is an épure. Anticipating upon section 4.3, since its vehicle is not a sum of binary stars, this anima is not a proof. We can however replace our épure with $[\![\frac{}{\boxed{R(\mathtt{l}\cdot x)}, \boxed{R(\mathtt{r}\cdot x)}}]\!] + [\![\frac{}{\boxed{S(\mathtt{l}\cdot x)}}]\!] + [\![\frac{}{\boxed{S(\mathtt{r}\cdot x)}, \boxed{T(x)}}]\!]$, which is quite a proof of of $\top$.

More generally, the sequent calculus rule :

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, \top}$$

which corresponds to the definition $\top := \exists X\, X$ makes sense in terms of épures: if $\mathcal{V} + \mathcal{M}$ is an épure with conclusions Γ, A, select an ordeal (in `yellow` and uncoloured) $\mathcal{O}$ for A. The sum $\mathcal{M} + \mathcal{O}$ normalises into some $\mathcal{N}$. If we replace all $\boxed{p_A(t)}$ with $\boxed{R(t)}$, all $p_A(t)$ with $\boxed{S(t)}$ and the uncoloured $p_A(x)$ with $\boxed{T(x)}$, then $\mathcal{V}$ and $\mathcal{N}$ respectively become $\mathcal{V}'$ and $\mathcal{N}'$; $\mathcal{V}' + \mathcal{N}'$ is an épure corresponding to a proof of $\vdash \Gamma, \top$.

4.3 Consistency

We already addressed the issue of consistency in our previous installment [5]: in order to be accepted as a proof, a vehicle must be a sum of binary stars $[\![t_i, u_i]\!]$. In the derealist case, an anima is accepted as a *proof* when it is an épure $\mathcal{V} + \mathcal{M}$ and its « objective » part, the vehicle $\mathcal{V}$, is a sum of binary stars.

Animæ which are not épures are sort of animist artifacts, flawed constructions mingling Object and Subject[6].

Consistency is therefore the existence of propositions without proofs. The typical example of such a proposition is the additive neutral $\mathbf{0}$ defined (with $\mathbf{0}(x) := p_\mathbf{0}(x), r(x) := p_\mathbf{0}(\mathtt{c}\cdot x), s(x) := p_\mathbf{0}(\mathtt{mag}(\mathtt{l}\cdot x)), t(x) := p_\mathbf{0}(\mathtt{mag}(\mathtt{r}\cdot x))$) by the three ordeals: $[\![\frac{r(x)}{}]\!] + [\![\frac{s(x), t(x)}{\mathbf{0}(x)}]\!]$ and $[\![\frac{s(x)}{}]\!] + [\![\frac{r(x), t(x)}{\mathbf{0}(x)}]\!]$ and $[\![\frac{t(x)}{\mathbf{0}(x)}]\!]$.

$\mathbf{0}$ admits as a « preproof » the anima $[\![\frac{}{\boxed{t(x)}}]\!] + [\![\frac{}{\boxed{r(x)}, \boxed{s(x)}}]\!]$ which complies with the ordeals for $\mathbf{0}$. However, there is no proof of $\mathbf{0}$, i.e., no épure $\mathcal{V} + \mathcal{G}$

[6]The *daimon* of ludics [2] proposed also a sort of animist « preproof »; its existence was dependent upon a polarisation of logic (negative/positive behaviours) a bit painful to handle. The approach through épures offers a simpler and better knitting.

complying with these ordeals. Indeed, any anima complying with them should use the rays $\boxed{r(x)}$, $\boxed{s(x)}$, $\boxed{t(x)}$; since it can contain neither $\llbracket \frac{}{\boxed{r(x)},\boxed{t(x)}} \rrbracket$ nor $\llbracket \frac{}{\boxed{s(x)},\boxed{t(x)}} \rrbracket$, it must contain $\llbracket \frac{}{\boxed{t(x)}} \rrbracket$ and also $\llbracket \frac{}{\boxed{r(x)},\boxed{s(x)}} \rrbracket$, which mixes the objective $\boxed{r(x)}$ and the subjective $\boxed{s(x)}$ and is thus animist.

This establishes the *consistency* of transcendental syntax: the absurdity has no « proof », i.e., harbours no épure.

By the way, the épure $\llbracket \frac{}{\boxed{R(x)},\boxed{r(x)}} \rrbracket + \llbracket \frac{}{\boxed{S(x)},\boxed{s(x)}} \rrbracket + \llbracket \frac{}{\boxed{T(x)},\boxed{t(x)}} \rrbracket$ is a proof of the sequent $\vdash \mathsf{T}, \mathbf{0}$.

4.4 Cut-elimination

The knitting *usine/usage*, i.e., cut-elimination, is an opportunity to see how normalisation works for épures in a simple case. Consider a cut $[C] := [\mathsf{T} \otimes \mathbf{0}]$; the cut is normalised as usual, by painting $p_{\mathsf{T}}(x)$ and $p_{\mathbf{0}}(x)$ in $\boxed{\texttt{green}}$ and adding the feedback (in `magenta`) $\llbracket \frac{p_{\mathsf{T}}(x), p_{\mathbf{0}}(x)}{} \rrbracket$.

The feedback $\llbracket \frac{p_{\mathsf{T}}(x), p_{\mathbf{0}}(x)}{} \rrbracket$ can be replaced with its specialisation, its « η-expansion » on the three sublocations $\mathtt{r}\cdot x, \mathtt{s}\cdot x, \mathtt{t}\cdot x$, i.e., with the sum $\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3$ with $\mathcal{T}_1 := \llbracket \frac{R(x), r(x)}{} \rrbracket$, $\mathcal{T}_2 := \llbracket \frac{S(x), s(x)}{} \rrbracket$ and $\mathcal{T}_3 := \llbracket \frac{T(x), t(x)}{} \rrbracket$.

Define $\mathcal{O} := \llbracket \frac{R(x), S(x)}{} \rrbracket$ and let $\mathcal{O}_1 := \llbracket \frac{T(x), s(x), t(x)}{p_C(x)} \rrbracket + \llbracket \frac{r(x)}{} \rrbracket$ and $\mathcal{O}_2 := \llbracket \frac{T(x), r(x), t(x)}{p_C(x)} \rrbracket + \llbracket \frac{s(x)}{} \rrbracket$ and $\mathcal{O}_3 := \llbracket \frac{T(x), t(x)}{p_C(x)} \rrbracket$. As well as the $\mathcal{P}_i$ by $\mathcal{P}_1 := \llbracket \frac{R(x), S(x), s(x), t(x)}{p_C(x)} \rrbracket + \llbracket \frac{r(x)}{} \rrbracket$, $\mathcal{P}_2 := \llbracket \frac{R(x), S(x), r(x), t(x)}{p_C(x)} \rrbracket + \llbracket \frac{s(x)}{} \rrbracket$ and $\mathcal{P}_3 := \llbracket \frac{R(x), S(x), t(x)}{p_C(x)} \rrbracket$.

The ordeals for C are the $\mathcal{O} + \mathcal{O}_i$ and $\mathcal{P}_1$ $(i = 1, \ldots, 3)$, the $\mathcal{P}_1$ being cancelling. If $\mathcal{Q}$ is an ordeal for Γ, an épure $\mathcal{V} + \mathcal{M}$ for $\Gamma, [C]$ must therefore comply with the $\mathcal{Q} + \mathcal{O} + \mathcal{O}_i$ and $\mathcal{Q} + \mathcal{P}_i$ $(i = 1, \ldots, 3)$, the $\mathcal{Q} + \mathcal{P}_i$ being cancelling.

If $\mathcal{V} + \mathcal{M}$ complies with the $\mathcal{Q} + \mathcal{O} + \mathcal{O}_i$, then $\mathcal{V} + \mathcal{M} + \mathcal{Q} + \mathcal{O}$ strongly normalises into a constellation $\mathcal{C}$ s.t. $\mathcal{C} + \mathcal{O}_i$ strongly normalises into $p_{\Gamma+C}(x)$ for $i = 1, 2, 3$. The free rays of $\mathcal{C}$ are $\boxed{T(x)}$, $\boxed{t(x)}$, $\boxed{r(x)}$ and $\boxed{s(x)}$, together with the $p_\Gamma(x)$. These rays are dispatched in various stars; considering $\mathcal{O}_1$ and $\mathcal{O}_2$, we see that $\boxed{r(x)}$ and $\boxed{s(x)}$ cannot be in the same star as $\boxed{T(x)}$ or $\boxed{t(x)}$; they must indeed share the same star $\mathcal{U} := \llbracket \frac{}{\boxed{r(x)},\boxed{s(x)}, p_{\Gamma'}(x)} \rrbracket$ for some $\Gamma' \subset \Gamma$.

$\mathcal{V} + \mathcal{M} + \mathcal{Q}$ strongly normalises into a constellation $\mathcal{D}$ s.t. $\mathcal{D} + \mathcal{O}$ normalises into $\mathcal{C}$. The constellation $\mathcal{D}$ involves additional free rays $\boxed{R(u_1)}$, $\boxed{S(u_1)}$,

$\ldots, \boxed{R(u_n)}, \boxed{S(u_n)}$; the star $\mathcal{U}$ is obtained from a subconstellation $\mathcal{D}' \subset \mathcal{D}$; now, if $\mathcal{D}' \neq \mathcal{U}$, one of the stars of $\mathcal{D}'$ must contain one of the two rays $\boxed{r(x)}, \boxed{s(x)}$ and one among the $\boxed{R(u_i)}, \boxed{S(u_i)}$. Now the cancelling ordeal $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) forbids any star involving $\boxed{R(u_i)}$ or $\boxed{S(u_i)}$ and $\boxed{s(x)}$ (resp. $\boxed{r(x)}$).

The normal form $\mathcal{D}$ of $\mathcal{V} + \mathcal{M} + \mathcal{Q}$ thus contains $\mathcal{U} := [\![\frac{}{\boxed{r(x)},\boxed{s(x)},p_{\Gamma'}(x)}]\!]$. Now, $\mathcal{D} + \mathcal{T}_1 + \mathcal{T}_2$ normalises into $\mathcal{D}'$ which is obtained from $\mathcal{D}$ by replacing $\mathcal{U} := [\![\frac{}{\boxed{r(x)},\boxed{s(x)},p_{\Gamma'}(x)}]\!]$ with $\mathcal{U}' := [\![\frac{R(x),S(x)}{p_{\Gamma'}(x)}]\!]$.

W.r.t. normalisation, $\mathcal{U}' := [\![\frac{R(x),S(x)}{p_{\Gamma'}(x)}]\!]$ and $\mathcal{T}_3 := [\![\frac{T(x),t(x)}{}]\!]$ play the same role as the stars $\mathcal{O} := [\![\frac{R(x),S(x)}{}]\!]$ and $\mathcal{O}_3 := [\![\frac{T(x),t(x)}{p_C(x)}]\!]$; the only difference being the choice of uncoloured rays ($p_{\Gamma'}(x)$ vs. $p_C(x)$). The constellation $\mathcal{D} + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3$ thus normalises into $[\![p_\Gamma(x)]\!]$. Since $\mathcal{V} + \mathcal{M} + \mathcal{Q}$ normalises to $\mathcal{D}$, we proved that $\mathcal{V} + \mathcal{M} + \mathcal{Q} + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3$ strongly normalises to $[\![p_\Gamma(x)]\!]$.

Like in ludics, cut-elimination works for general animæ. Now remember that the normal form of an épure is an épure; and that the normal form of a vehicle made of binary stars is still made of binary stars; therefore, provided normalisation converges, the normal form of a proof remains a proof. This means that our notion of consistency is a real, deductive one, not a paraconsistent, non deductive doohickey. In fact, we can define truth as the existence of a proof in the sense of section 4.3. Which is definitely more satisfactory than the Tarskian approach — truth as the fact of being true.

5 First order predicate calculus

We content ourselves with the usine aspects, skipping the adequation with usage.

5.1 First order quantifiers

Usual predicate calculus makes use of variables or terms as if they were natural things to consider. Our first problem is to find the conditions of possibility for first order variables; it is indeed very difficult to say what they stand for. Computer science however provides us with a stisfactory answer: a variable is an address, i.e., a location. In particular, it bears no relation with the variables $x, y, z, \ldots$ used in rays, stars and constellations. In order to avoid confusion, I therefore propose to use $\alpha, \beta, \gamma, \ldots$ for variables and $\sigma, \tau, \rho, \ldots$ for individuals defined as multiplicative expressions (section 2.3).

Our goal is to interprete first order predicate calculus based upon literals $\sigma = \tau$, $\sim(\sigma = \tau)$ by means of the connectives $\otimes, \parr, \oslash, \ltimes, \&, \oplus, \top, \mathbf{0}$ and the quantifiers $\forall\alpha$, $\exists\alpha$, $\forall\beta$, $\exists\beta$, $\forall\gamma$, $\exists\gamma$, etc.

Sequents now take the dependent form $\vdash^{\boldsymbol{\beta}} \Gamma$, where $\boldsymbol{\beta} = \beta_1, \ldots, \beta_n$ is a list of variables: those which are *declared*, i.e., allowed to occur freely in Γ. The rules for quantification are the following:

$$\frac{\vdash^{\boldsymbol{\beta},\boldsymbol{\alpha}} \Gamma, A}{\vdash^{\boldsymbol{\beta}} \Gamma, \forall\alpha A} \qquad\qquad \frac{\vdash^{\boldsymbol{\beta}} \Gamma, A[\sigma/\alpha]}{\vdash^{\boldsymbol{\beta}} \Gamma, \exists\alpha A}$$

The $\forall$ rule removes α (supposedly not occurring in Γ) from the stock of declared variables $\boldsymbol{\beta}, \alpha$; in the $\exists$ rule, the variables occurring in σ must be chosen among $\boldsymbol{\beta} = \beta_1, \ldots, \beta_n$. In particular, since there is no closed individual, one cannot prove any sequent of the form $\vdash \exists\alpha_1 A_1, \ldots, \exists\alpha_k A_k$; as a consequence, $\forall\alpha A \Rightarrow \exists\beta A[\beta/\alpha]$ fails.

5.2 Explicit substitution

We define $p_\alpha, p_{\tilde{\alpha}}, p_A$ from $p_{\forall\alpha A}$ (or $p_{\exists\alpha A}$) by[7]:

$$p_\alpha(x) := p_{\forall\alpha A}(\mathtt{r}\cdot x), p_{\tilde{\alpha}}(x) := p_{\forall\alpha A}(\mathtt{l}\cdot x), p_A(x) := p_{\forall\alpha A}(\mathtt{c}\cdot x)$$

$$p_\alpha(x) := p_{\exists\alpha A}(\mathtt{l}\cdot x), p_{\tilde{\alpha}}(x) := p_{\exists\alpha A}(\mathtt{r}\cdot x), p_A(x) := p_{\exists\alpha A}(\mathtt{c}\cdot x)$$

We must now face the conditions of possibility of variables. Which is definitely more demanding than the flippant axiomatic approach for which a variable is hardly more than a symbol styled « variable ».

- A variable α is either universal or existential. The axiomatic variable used in the dubious proof of $\forall\alpha A \Rightarrow \exists\alpha A$ belongs to neither category: it has therefore been excluded.

- A variable occurs under the dual form $\alpha/\tilde{\alpha}$.

- α occurs both in the prefix $\forall\alpha$ (or $\exists\alpha$) and various occurrences in the body A; the most important occurrence is the prefix, related to the other ones by an explicit substitution, for which a special sublocation $x\mathtt{c}y$ is used.

- Explicit substitution is not the sole use of the prefix; another sublocation $x\mathtt{d}y$ is needed. In the universal case, it takes care of the occurrences of α in existential witnesses; in the existential case, of the limited complementarity between the two parts $\mathcal{G}, \tilde{\mathcal{G}}$ of the mould.

Whether quantified existentially or universally, α and $\tilde{\alpha}$ are expected to occur in A. Let us concentrate on an occurrence α_i of α. If $B := \exists\alpha A$ (or $\forall\alpha A$), then $q_{\alpha_i}(x)$ is of the form $p_B(\mathtt{c}\cdot(\mathtt{t}_1\cdot(\mathtt{t}_2 \ldots (\mathtt{t}_m\cdot(\mathtt{g}\cdot x)) \ldots)))$, where the $\mathtt{t}_i$ are constants $\mathtt{l}, \mathtt{r}, \mathtt{c}, \ldots$. Moreover, the ordeals for A actually use $q_{\alpha_i}((\ldots(x \cdot y_n) \ldots y_2) \cdot y_1)$ where the variables y_i take into account exponentiations.

In the traditional formulation of existential quantification, a substitution must be performed, q_{α_i} becoming some q_σ. Consistently with transcendentalism, this operation cannot be the deed of some Tarskian demon proceeding from

[7] Observe tha swapping between p_α and $p_{\tilde{\alpha}}$.

the sky: it must be expressed as an *explicit substitution*, to mention an important contribution [1] of the same Curien to whom this paper is dedicated. What is substituted will be localised at the address p_α and delocating LEGO bricks to and fro p_{α_i} will be provided. With the shorthand *abc* for $a\cdot(b\cdot c)$, e.g., $x\mathtt{c}y_3y_2y_1$ for $x\cdot(\mathtt{c}\cdot(y_3\cdot(y_2\cdot y_1)))$:

$$[\![\frac{q_\alpha(\mathtt{grn}(x\mathtt{c}\mathtt{t}_1\ldots\mathtt{t}_m y_n\ldots y_1))}{\boxed{q_{\alpha_i}(xy_n\ldots y_1)}}]\!]+[\![\frac{p_{\alpha_i}(xy_n\ldots y_1)}{\boxed{q_\alpha(\mathtt{mag}(x\mathtt{c}\mathtt{t}_1\ldots\mathtt{t}_m y_n\ldots y_1))}}]\!]$$

If α_i were a real atom, we would write the direct connection $[\![\frac{p_{\alpha_i}(xy_n\ldots y_1)}{\boxed{q_{\alpha_i}(xy_n\ldots y_1)}}]\!]$; instead, we transit through q_α. Observe the internal use of $\mathtt{grn}(\cdot)$ and $\mathtt{mag}(\cdot)$ which make our intermediate locations *subjective*. These two stars are part of the ordeals for the quantifiers $\exists\alpha$ and $\forall\alpha$. Ditto for occurrences $\tilde{\alpha}_j$ of $\tilde{\alpha}$.

5.3 Universal quantification

The passage from $\vdash^{\boldsymbol{\beta},\alpha}\Gamma, A$ to $\vdash^{\boldsymbol{\beta}}\Gamma,\forall\alpha A$ does not affect the épure.

The ordeal for $\forall\alpha A$ is made of the star $[\![\frac{q_A(x)}{\boxed{q_{\forall\alpha A}(x)}}]\!]$, of the explicit substitutions (section 5.2) for the α_i and $\tilde{\alpha}_j$, as well as:

$$[\![\frac{q_\alpha(\mathtt{grn}(x\mathtt{d}y))}{p_\alpha(\mathtt{grn}(x\mathtt{d}y))}]\!]+[\![\frac{p_\alpha(\mathtt{mag}(x\mathtt{d}y))}{\boxed{q_\alpha(\mathtt{mag}(x\mathtt{d}y))}}]\!]+[\![\frac{q_{\tilde{\alpha}}(\mathtt{grn}(x\mathtt{d}y))}{p_{\tilde{\alpha}}(\mathtt{grn}(x\mathtt{d}y))}]\!]+[\![\frac{p_{\tilde{\alpha}}(\mathtt{mag}(x\mathtt{d}y))}{\boxed{q_{\tilde{\alpha}}(\mathtt{mag}(x\mathtt{d}y))}}]\!]$$

These four stars are used to connect moulds (*infra*) depending upon $\alpha,\tilde{\alpha}$ with our switches. Observe the use of a constant $\mathtt{d}\neq\mathtt{c}$: this is to keep a clear distinction between the occurrences of α in the A of $\forall\alpha A$ (handled by $\mathtt{c}$) and the occurrences of α used in existential witnesses (handled by $\mathtt{d}$).

Moreover, a ternary switch accounts for the generic nature of α:

$$\forall_\otimes\texttt{:}\ [\![\frac{q_\alpha(\mathtt{mag}((\mathtt{l}x)y)), q_\alpha(\mathtt{mag}((\mathtt{r}x)y))}{\boxed{q_\alpha(\mathtt{grn}(xy))}}]\!]+[\![\frac{q_{\tilde{\alpha}}(\mathtt{mag}((\mathtt{l}x)y))}{\boxed{q_{\tilde{\alpha}}(\mathtt{grn}(xy))}}]\!]+[\![\frac{q_{\tilde{\alpha}}(\mathtt{mag}((\mathtt{r}x)y))}{\boxed{q_{\tilde{\alpha}}(\mathtt{grn}(xy))}}]\!].$$

$$\forall_{\invamp}\texttt{:}\ [\![\frac{q_\alpha(\mathtt{mag}((\mathtt{l}x)y))}{\boxed{q_\alpha(\mathtt{grn}(xy))}}]\!]+[\![\frac{q_\alpha(\mathtt{mag}((\mathtt{r}x)y))}{\boxed{q_\alpha(\mathtt{grn}(xy))}}]\!]+[\![\frac{q_{\tilde{\alpha}}(\mathtt{mag}((\mathtt{l}x)y)), q_{\tilde{\alpha}}(\mathtt{mag}((\mathtt{r}x)y))}{\boxed{q_{\tilde{\alpha}}(\mathtt{grn}(xy))}}]\!].$$

$$\forall_{Id}\texttt{:}\ [\![\frac{q_\alpha(\mathtt{mag}(xy))}{\boxed{q_\alpha(\mathtt{grn}(xy))}}]\!]+[\![\frac{q_{\tilde{\alpha}}(\mathtt{mag}(xy))}{\boxed{q_{\tilde{\alpha}}(\mathtt{grn}(xy))}}]\!]+[\![\frac{q_\alpha(\mathtt{mag}(x'y))}{\boxed{q_\alpha(\mathtt{grn}(x'y))}}]\!]+[\![\frac{q_{\tilde{\alpha}}(\mathtt{mag}(x''y))}{\boxed{q_{\tilde{\alpha}}(\mathtt{grn}(x''y))}}]\!],$$

except if $x=x'$ or $x=x''$ (explanation *infra*).

The position $\forall_\otimes$ (resp. $\forall_{\invamp}$) corresponds to the choices $\alpha=\otimes,\tilde{\alpha}=\invamp$ (resp. $\alpha=\invamp,\tilde{\alpha}=\otimes$); both ensure the synchronisation between α and $\tilde{\alpha}$. The third position $\forall_{Id}$ is there to exclude a practical joke, namely that an épure making use of the sole sublocations $(\mathtt{l}x)y$ and $(\mathtt{r}x)y$ used in the first two switchings. Moreover, in the spirit of the connective $\ltimes$ [4], the stars are duplicated; the duplicated star $\llbracket \frac{q_\alpha(\mathtt{mag}(x'y))}{\boxed{q_\alpha(\mathtt{grn}(x'y))}} \rrbracket$ comes with coherences w.r.t. $\llbracket \frac{q_\alpha(\mathtt{mag}(xy))}{\boxed{q_\alpha(\mathtt{grn}(xy))}} \rrbracket$ and $\llbracket \frac{q_A(x)}{\boxed{q_{\forall\alpha A}(x)}} \rrbracket$: in both cases $x' \neq x$. Ditto with $\llbracket \frac{q_{\tilde{\alpha}}(\mathtt{mag}(x''y))}{\boxed{q_{\tilde{\alpha}}(\mathtt{grn}(x''y))}} \rrbracket$.

5.4 Moulds

An existential variable, say α, is bound to receive a value, namely a gabarit $\mathcal{G}$; its negation $\tilde{\alpha}$ will simultaneously receive the value $\tilde{\mathcal{G}}$. Since we are dealing with « first order », $\mathcal{G}$ must correspond to a multiplicative individual σ, i.e., depend upon universal variables — typically those declared in our sequent calculus. $\mathcal{G}$ and $\tilde{\mathcal{G}}$, delocated in p_α, $p_{\tilde{\alpha}}$ will form a *mould*, i.e., an existential witness. These moulds are always *subjective*: derealism is a typical product of existential quantification. Remember that an individual is a multiplicative expression depending upon variables (indeed universal ones) $\beta, \beta', \beta'', \ldots$ (but not their negations), possibly with repetitions. For reasons of legibility, I will concentrate on an example, namely that of $\sigma := \beta \otimes \beta$.

$$\mathcal{G} := \llbracket \frac{\boxed{p_\beta(\mathtt{grn}(x\mathtt{dl}y))}, \boxed{p_\beta(\mathtt{grn}(x\mathtt{dr}y))}}{\boxed{p_\alpha(\mathtt{grn}(xy))}} \rrbracket + \llbracket \frac{\boxed{p_\alpha(\mathtt{mag}(xy))}}{\boxed{p_\beta(\mathtt{mag}(x\mathtt{d}y))}} \rrbracket$$

As future component of an épure, $\mathcal{G}$ has been painted $\boxed{\mathtt{blue}}$. Observe the use of $\mathtt{d}$ in $x\mathtt{dl}y$ (instead of $x\mathtt{l}y$).

Consistently with [5], the gabarit corresponding to $\tilde{\sigma}$ is defined as

$$\tilde{\mathcal{G}} := (\frac{3}{5} \cdot \Upsilon_L \overset{\smile}{+} \frac{4}{5} \cdot \Upsilon_R) + \llbracket \frac{\boxed{p_{\tilde{\alpha}}(\mathtt{mag}(xy))}}{\boxed{p_{\tilde{\beta}}(\mathtt{mag}(x\mathtt{d}y))}} \rrbracket$$

with:

$$\Upsilon_L := \quad \llbracket \frac{\boxed{p_{\tilde{\beta}}(\mathtt{grn}(x\mathtt{dl}y))}}{\boxed{p_{\tilde{\alpha}}(\mathtt{grn}(xy))}} \rrbracket + \llbracket \frac{\boxed{p_{\tilde{\beta}}(\mathtt{grn}(x\mathtt{dr}y))}}{} \rrbracket$$

$$\Upsilon_R := \quad \llbracket \frac{\boxed{p_{\tilde{\beta}}(\mathtt{grn}(x\mathtt{dr}y))}}{\boxed{p_{\tilde{\alpha}}(\mathtt{grn}(xy))}} \rrbracket + \llbracket \frac{\boxed{p_{\tilde{\beta}}(\mathtt{grn}(x\mathtt{dl}y))}}{} \rrbracket$$

The *mould* $\mathcal{M}_\sigma$ corresponding to the individual σ is defined as the sum $\mathcal{G} + \tilde{\mathcal{G}}$ of the two constellations just defined.

A gabarit, originally coloured in magenta/$\boxed{\mathtt{green}}$ has been rewritten in blue, the original colours becoming $\mathtt{mag}(\cdot)$ and $\mathtt{grn}(\cdot)$. However, we didn't take care of the second pair of colours (originally cyan/$\boxed{\mathtt{red}}$) used for mosaics [5]. If these colours are rendered by $\mathtt{cya}(\cdot)$ and $\mathtt{red}(\cdot)$, our mould is therefore expected to use rays of the form $\boxed{p_\alpha(\mathtt{red}(xy))}, \boxed{p_\alpha(\mathtt{cya}(xy))}, \boxed{p_{\tilde{\alpha}}(\mathtt{red}(xy))}, \boxed{p_{\tilde{\alpha}}(\mathtt{cya}(xy))}$.

5.5 Existential quantification

If $\vdash^{\boldsymbol{\beta}} \Gamma, \exists\alpha A$ has been obtained from $\vdash^{\boldsymbol{\beta}} \Gamma, A[\sigma/\alpha]$, then its proof is obtained by summing up the proof $\mathcal{V} + \mathcal{M}$ of the premise and the mould $\mathcal{M}_\sigma$ corresponding to σ and located at $p_\alpha, p_{\tilde\alpha}$.

The ordeal for $\exists\alpha A$ makes use of the following constellations:

$$[\![\frac{p_\alpha(\mathtt{grn}(x\mathtt{c}y))}{\boxed{q_\alpha(\mathtt{grn}(x\mathtt{c}y))}}]\!] + [\![\frac{q_\alpha(\mathtt{mag}(x\mathtt{c}y))}{p_\alpha(\mathtt{mag}(x\mathtt{c}y))}]\!] + [\![\frac{p_{\tilde\alpha}(\mathtt{grn}(x\mathtt{c}y))}{\boxed{q_{\tilde\alpha}(\mathtt{grn}(x\mathtt{c}y))}}]\!] + [\![\frac{q_{\tilde\alpha}(\mathtt{mag}(x\mathtt{c}y))}{p_{\tilde\alpha}(\mathtt{mag}(x\mathtt{c}y))}]\!]$$

which naturally bridges with the explicit substitutions of section 5.2. And

$$[\![\frac{q_\alpha(\mathtt{cya}(xy))}{p_\alpha(\mathtt{cya}(xy))}]\!] + [\![\frac{p_\alpha(\mathtt{red}(xy))}{\boxed{q_\alpha(\mathtt{red}(xy))}}]\!] + [\![\frac{q_{\tilde\alpha}(\mathtt{cya}(xy))}{p_\alpha(\mathtt{cya}(xy))}]\!] + [\![\frac{p_{\tilde\alpha}(\mathtt{red}(xy))}{\boxed{q_\alpha(\mathtt{red}(xy))}}]\!]$$

which takes care of the mosaic aspects [5] of the mould.

The four-ary switch:

$$\exists_{Ldd}\textbf{:}\ [\![\frac{q_A(x), p_\alpha(\mathtt{grn}(x\mathtt{dd}))}{\boxed{q_{\exists\alpha A}(x)}}]\!] + [\![\frac{p_{\tilde\alpha}(\mathtt{grn}(x\mathtt{dd}))}{}]\!]$$

$$\exists_{Ldx}\textbf{:}\ [\![\frac{q_A(x), p_\alpha(\mathtt{grn}(x\mathtt{d}x))}{\boxed{q_{\exists\alpha A}(x)}}]\!] + [\![\frac{p_{\tilde\alpha}(\mathtt{grn}(x\mathtt{d}x))}{}]\!]$$

$$\exists_{Rdd}\textbf{:}\ [\![\frac{q_A(x), p_{\tilde\alpha}(\mathtt{grn}(x\mathtt{dd}))}{\boxed{q_{\exists\alpha A}(x)}}]\!] + [\![\frac{p_\alpha(\mathtt{grn}(x\mathtt{dd}))}{}]\!]$$

$$\exists_{Rdx}\textbf{:}\ [\![\frac{q_A(x), p_{\tilde\alpha}(\mathtt{grn}(x\mathtt{d}x))}{\boxed{q_{\exists\alpha A}(x)}}]\!] + [\![\frac{p_\alpha(\mathtt{grn}(x\mathtt{d}x))}{}]\!]$$

together with the star $[\![\frac{}{p_\alpha(\mathtt{mag}(x\mathtt{d}y)), p_{\tilde\alpha}(\mathtt{mag}(x\mathtt{d}y))}]\!]$ (a sort of identity link restricted to the sublocation $x\mathtt{d}y$), tests what can be checked of the complementarity between the two parts $\mathcal{G}, \tilde{\mathcal{G}}$ of the mould: as explained in section 6.1 below, this only ensures a *partial* complementarity, a sort of *dinaturality.* The choice and $\mathtt{dd}/\mathtt{d}x$ in the switch compensates the impossibility to use y; a similar gimmick was used in [4] to handle $\oslash$.

5.6 Predicate variables

As we explained in [4], propositional constants are a mistake — of the same sort as non generic variables; ditto for predicate constants. Predicates are indeed universally quantified variables; but the quantifier is kept implicit. What follows is rather sketchy: the precise treatment would involve too much red tape.

In presence of an injective pairing, unary predicates are enough. A predicate $P(\cdot,\cdot)$ is a lacunary structure with two « holes » to be filled with an individual σ and its negation $\tilde\sigma$: $P(\sigma,\tilde\sigma)$, together with a *root*, the two holes being disjoint sublocations of the root. They do occur in atoms as $P(\sigma,\tilde\sigma)$ and $\tilde Q(\tilde\tau,\tau)$.

The main synthetic problem is to ensure that identity links between atoms do relate P with $\tilde P$ (and not with Q or $\tilde Q$), hence $P(\sigma,\tilde\sigma)$ with $\tilde P(\tilde\tau,\tau)$; moreover, that σ $(=\tau)$ relates with $\tilde\tau$ and $\tilde\sigma$ with τ.

What is ensured by a switching of the implicit universal quantification $\forall P$: either $P(\sigma, \tilde{\sigma})$ is interpreted by the binary connective $\sigma \otimes \tilde{\sigma}$ (in which case $\tilde{P}(\tilde{\sigma}, \sigma)$ becomes $\tilde{\sigma} \parr \sigma$), or $P(\sigma, \tilde{\sigma})$ is interpreted by the binary connective $\sigma \parr \tilde{\sigma}$ (in which case $\tilde{P}(\tilde{\sigma}, \sigma)$ becomes $\tilde{\sigma} \otimes \sigma$). Since the switchings of the various $P, Q, \ldots$ are independent, the only possibility is to relate $P(\sigma, \tilde{\sigma})$ with $\tilde{P}(\tilde{\tau}, \tau)$; the relation between σ and τ is taken care of by the general handling of first order variables.

Unfortunately, our usine justifies $(\sigma = \tau \otimes \sigma = \tau) \multimap (P(\sigma, \tilde{\sigma}) \multimap P(\tau, \tilde{\tau}))$, i.e., something like linearity of P. If we keep in mind that $\forall P$ should eventually be cut with $\exists P$ which refers to arbitrary moulds, we get the definitely faulty $(\sigma = \tau \otimes \sigma = \tau) \multimap (A[\sigma, \tilde{\sigma}] \multimap A[\tau, \tilde{\tau}])$. Indeed, predicate variables should only enjoy the weaker $\sigma = \tau \Rightarrow (P(\sigma, \tilde{\sigma}) \multimap P(\tau, \tilde{\tau}))$. Instead of switching $P/\tilde{P}$ with linear connectives $\otimes/\parr$ (or $\parr/\otimes$), I propose to switch them with the non linear $\olessthan$ and $\ltimes$: either $P(\sigma, \tilde{\sigma})$ is interpreted by the binary connective $\sigma \olessthan \tilde{\sigma}$ (in which case $\tilde{P}(\tilde{\sigma}, \sigma)$ becomes $\tilde{\sigma} \ltimes \sigma$), or $P(\sigma, \tilde{\sigma})$ is interpreted by the binary connective $\sigma \ltimes \tilde{\sigma}$ (in which case $\tilde{P}(\tilde{\sigma}, \sigma)$ becomes $\tilde{\sigma} \olessthan \sigma$).

6 Epidictics

6.1 Partiality

The derealism at work in quantification is reponsible for the loss of certainty: logic is no longer *apodictic*, an expression whose etymology is « proven ». This phenomenon can be ascribed to the subjective — synthetic — component of the proof, the *mould*. The mould $\mathcal{M}$ usually comes in two parts $\mathcal{G}, \tilde{\mathcal{G}}$ supposedly the negation of each other: $\tilde{\mathcal{G}} = \sim\mathcal{G}$. I use the graphism $\sim\mathcal{G}, \sim\tilde{\mathcal{G}}$ to speak of the « actual » negations, which may differ from $\tilde{\mathcal{G}}, \mathcal{G}$.

The problem originates from the fact that negation (and implication, based on an implicit use of negation) are not genuine connectives: we cannot really *construct* $\sim A$ from A. This is obvious when we look at proof-nets: the negation of the conjunction $\otimes$ is *given* by $\parr$ and vice versa; however, it is possible to show that the two connectives negate each other. Now given arbitrary $\mathcal{G}$ and $\tilde{\mathcal{G}}$, can we still check that they negate each other? Indeed, by writing an identity link between the two, it is possible to check whether or not $\vdash \mathcal{G}, \tilde{\mathcal{G}}$ holds: in other terms whether $\sim\tilde{\mathcal{G}} \multimap \mathcal{G}$ (equivalently, $\sim\mathcal{G} \multimap \tilde{\mathcal{G}}$). If $\mathcal{G}$ and $\tilde{\mathcal{G}}$ actually negate each other, we just written the familiar *identity principle*. The component $[\![\, \overline{p_\alpha(\texttt{mag}(x\texttt{d}y)), p_{\tilde{\alpha}}(\texttt{mag}(x\texttt{d}y))} \,]\!]$ of ordeals actually checks that $\sim\tilde{\mathcal{G}} \multimap \mathcal{G}$.

But what is missing is the cut rule, i.e., the converse $\vdash \sim\mathcal{G}, \sim\tilde{\mathcal{G}}$, in other terms the implication $\mathcal{G} \multimap \sim\tilde{\mathcal{G}}$ (equivalently, $\tilde{\mathcal{G}} \multimap \sim\mathcal{G}$). Eighty-odd years of cut-elimination theorems taught us that cut-elimination is by no means a straightforward notion: it embodies, in some sense, all foundational issues. Therefore the converse cannot be checked. This opens the possibility of a sort of schizophrenia: the mould consists of a laxist form ($\mathcal{G}$) and a strict one ($\sim\tilde{\mathcal{G}}$).

This situation was first noticed by Schütte [7], who interpreted second order logic by *partial valuations* — i.e., three-valued models —: a strict version, being true, vs. a laxist one, being unfalse. The cut rule required the equivalence between the two versions, hence the totality of the « valuation ». Cut-elimination was thus reduced to a matter of completion of a partial valuation into a total one. Schütte's approach remains the most direct explanation of a possible gap between $\mathcal{G}$ « not false » and $\sim\tilde{\mathcal{G}}$ « true ». But also the bleakest: the third value **u** — « u » for undefined, but also useless, unfit — has a propensity to phagocyte the real ones **t**,**f**, to the point that most formulas take this very unexciting value. A much better handling of the same is to be found in *dinaturals*, another shizophrenic approach in which a morphism from the strict to the laxist version is provided. The « hexagonal » diagrams (see, e.g., [3]) express, through a want of compositionality, the possibility of a mimatch usine/usage; this in a more civilised way than the value « unfit ».

6.2 Quantification

We reached the conclusion that quantification is a matter of propositional logic. But there are, traditionally speaking, various quantifications, first, second and even higher order ones. My claim is that there is but one quantification dealing with *partial* moulds, i.e., moulds enjoying $\sim\tilde{\mathcal{G}} \multimap \mathcal{G}$. The problem being that those partial moulds do not guarantee l'usage.

The situation is rather similar to that of naive set theory: the naive comprehension axiom can be handled by means of *partial* sets $a^s \subset a^l$ — a strict and a laxist version. Typically, if $a := \{x; x \notin x\}$, then $a \in a^l \setminus a^s$. The principles of ZF are ways of ensuring totality, i.e., $a^s = a^l$. But prior to ZF, there were also type theories dedicated to the same problem.

Epidictics[8] is the fact of claiming the totality of specific moulds. The easiest way is to devise specific classes of moulds whose totality we ascertain: since several choices can can coexist, this explains the apparent plurality of quantifiers.

I essentially considered the « first order » case $\forall\alpha, \exists\alpha$: the epidictic restriction consists in choosing our moulds among multiplicative individuals. The matching between σ and $\tilde{\sigma}$, although external, is rather unproblematic.

I didn't quite consider second order quantification, except in the castrated forms $\forall X\, X, \exists X\, X$: due to the absence of $\sim X$, epidictics hardly matters. Of course, second and higher order quantifications should be investigated, keeping in mind that we can fine tune our quantifiers so as to, say, represent weak forms of arithmetic recurrence.

But the real problem at stake is the use of a single, untyped quantifier, in the spirit of set theory. We know that hastily written epidictic principles — like the first version of Martin-Löf's type theory — may fail, i.e., happen to be partial. The question indeed is not quite that of finding a universal epidictics: incompleteness forbids us forever to answer this question; moreover, no consensus on

[8]Expression created in analogy with « apodictic »; the epidictic style is the style of excessive praise, typical of obituaries: « The greatest man who ever lived ».

the right principles may be expected, witness the literature on « predicativity ». The real problem is to understand the *structure* of the epidictic layer, if any. For instance, Martin-Löf's type theory [6] involves epidictic judgments of the kind « A is a type ». Something is proven there, but what? We would like to know the transcendental status of these proofs, in particular determine in which way this sort of epidictic judgment is part of the logical process.

At the present moment, epidictics is but a name on a blank area of the logical charts; hence a sort of new frontier for logic.

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